and define the following:

$$A = \sum_{k=1}^{7} a_k z^k$$

$$A + A^3/3! + A^5/5! + \ldots = \sum_{k=1}^{\infty} p_n z^n$$

$$A^2/2! + A^4/4! + \ldots = \sum_{n=2}^{\infty} q_n z^n$$

$$r_0 = 1$$

$$r_1 = 0 \qquad s_1 = a_1$$

$$r_2 = \tfrac{1}{2}a_1^2 \qquad s_2 = a_2$$

$$r_3 = a_1 a_2 \qquad s_3 = a_3 + a_1^3/6$$

Then, the coefficient g_j of the series expansion given by Eq. (C1) is

$$g_j = \sum_{n=0}^{m} r_n p_{n+j} - \sum_{n=1}^{m} s_n q_{n+j} \quad j = 1,2,\ldots,7 \tag{C2}$$

where m is the largest integer equal or less than $\frac{1}{2}(7 - j)$. This gives

$$g_1 = a_1 - \tfrac{1}{2}a_1^3 + \tfrac{1}{2}a_1^2(a_3 + a_1^3/6) - a_2^2 a_1 + a_1 a_2(a_4 + \tfrac{1}{2}a_1^2 a_2) - (a_3 + a_1^3/6)(\tfrac{1}{2}a_2^2 + a_1 a_3 + a_1^4/24)$$

$$g_2 = a_2 - a_1^2 a_2 + \tfrac{1}{2}a_1^2(a_4 + \tfrac{1}{2}a_1^2 a_2) - a_2(\tfrac{1}{2}a_2^2 + a_1 a_3 + a_1^4/24) \tag{C3}$$

$$g_3 = a_3 + a_1^3/6 - a_1(\tfrac{1}{2}a_2^2 + a_1 a_3 + a_1^4/24) + \tfrac{1}{2}a_1^2(a_5 + a_1 a_2^2/6 + \tfrac{1}{2}a_1^2 a_3 + a_1^5/120) - a_2(a_1 a_4 - a_2 a_3)$$

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Influence of Damping on the Dynamic Stability of Spherical Caps under Step Pressure Loading

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An explicit numerical procedure for solving the nonlinear axisymmetric shell equations of Sanders for dynamic loadings is presented. Dynamic buckling pressures are determined as a function of the geometric parameter λ for several values of damping.

Introduction

The transient response of thin shells to dynamic loading conditions is a subject of considerable practical interest. It has thus attracted the attention of numerous researchers in structural mechanics. Until quite recently, attention had largely been focused on linear problems, but the success of numerical procedures in the area of nonlinear static problems naturally encouraged an attack on the problem of dynamic buckling of thin shells. As a result, several interesting articles have recently appeared[1–4] which present reliable techniques and accurate results. These authors employ finite-difference procedures (Stricklin and Martinez[4] use a finite-element approach) to solve the nonlinear partial differential equations governing the axisymmetric large deflections of spherical caps. They also chose an implicit finite-difference scheme rather than an explicit technique. Because of the nonlinear character of the governing equations, the selection of an implicit procedure requires that, to advance the solution, at each time step one must solve a set of nonlinear algebraic equations whose order is equal to a multiple of the number of mesh points in space. Thus, an iterative procedure is required, and, as Archer and Lange[1] observe, unless one is chosen with at least a quadratic rate of convergence, the number of iterations required increases dramatically. Archer and Lange were the first to employ a completely numerical approach to the nonlinear shell problems. These authors

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indicate that they abandoned explicit procedures as impractical because of difficulties associated with numerical stability. Explicit numerical procedures have been used to obtain the dynamic response of elastic columns to time dependent axial loads.[5,6]

Our objective in the present paper is twofold. We demonstrate, first, that the use of an explicit finite difference procedure in problems of this type is highly efficient, requiring only about one-third the computer time needed for the same accuracy by an implicit procedure. Since dynamic problems typically require an order of magnitude more computation than their static counterparts, there is a premium on efficiency when one needs to consider a variety of loading and boundary conditions.

Our second objective is to examine the role of damping in dynamic elastic stability of spherical caps. This is of interest because the undamped dynamic buckling load for a suddenly applied uniform pressure is, roughly, one-half of the static value. A recent article[7] on the role of damping in shallow arches contains the curious result that, if even an extremely small amount of damping is present, there exist no significant differences between statically and dynamically applied buckling loads. The result is then sweepingly generalized (without detailed analysis) to rings, cylinders, and spheres. The results obtained in the present study are in marked contrast to the assertions made by Hegemier and Tzung.[7]

Basic Equations

The governing equations for this study were taken to be those presented by Sanders.[8] When specialized to an elastic spherical shell, these may be reduced to the following pair of coupled nonlinear equations for the meridional and normal displacement u and w,

$$\begin{aligned} w''' + \cot\phi w'' - [\cot^2\phi + \nu + \xi(1+\nu)]w' - \\ \xi[u'' + \cot\phi u' + (\nu + \cot^2\phi)u] - \\ (R^3/D)\,\{Eh[(\beta^2/2)' + (\cot\phi)(1-\nu)\beta^2/2]/(1-\nu^2) - \\ \beta N_\phi\} = -R^4(\rho h\ddot{u} + \bar{\lambda}\ddot{u} + ku)/D \quad (1) \end{aligned}$$

$$\begin{aligned} w'''' + (2\cot\phi)w''' - (1 + \nu + \cot^2\phi)w'' + \\ (2 - \nu + \cot^2\phi)\cot\phi w' + 2\xi(1+\nu)w - u''' - \\ 2\cot\phi u'' + [1 + \nu + \cot^2\phi + \xi(1+\nu)]u' - \\ [2 - \nu + \cot^2\phi - \xi(1+\nu)]\cot\phi u + \\ R^3\{Eh(1+\nu)\beta^2/[2(1-\nu^2)] + (\beta N_\phi)' + \cot\phi N_\phi\beta\}/D = \\ -R^4[\rho h\ddot{w} + \bar{\lambda}\dot{w} + kw + p(\phi, t)]/D \quad (2) \end{aligned}$$

where ( )′ refers to differentiation with respect to the angle ϕ (see Fig. 1), (˙) refers to differentiation with respect to time, R is the radius of the sphere, h is the shell thickness, E, ν, and ρ are Young's modulus, Poisson's ratio, and density of the shell material, and $p(\phi,t)$ is the component of applied load normal to the middle surface. We also define $D = Eh^3/12(1 - \nu^2)$ and $\xi = 12R^2/h^2$. $\bar{\lambda}$ and K are the coefficients of the damping and restoring force terms. In Eqs. (1) and (2), the rotation β is expressible in terms of u and w as $\beta = (u - w')/R$, whereas the meridional stress resultant N_ϕ is given by

$$N_\phi = Eh[u' + w + \nu(\cot\phi u + w)]/R(1-\nu^2) + Eh\beta^2/[2(1-\nu^2)]$$

This expression clearly leads to terms of third degree in β in Eqs. (1) and (2). These would normally be thought to be negligible, but as we shall indicate later it turns out that predictions of dynamic buckling are very sensitive to their presence.

Equations (1) and (2) must be supplemented by appropriate boundary and initial conditions. We shall consider shells initially at rest, with zero velocity so that

$$u(\phi,0) = w(\phi,0) = \dot{u}(\phi,0) = \dot{w}(\phi,0) = 0$$

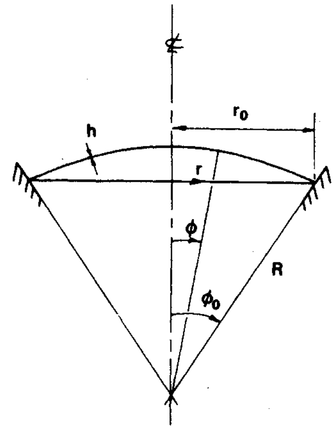

Fig. 1 Geometry.

For spherical shells closed at the apex, we can insure finite conditions by requiring

$$u(0,t) = w'(0,t) = 0$$

whereas, e.g., for clamped edges at $\phi = \phi_0$, we have

$$u(\phi_0,t) = w'(\phi_0,t) = w'(\phi_0,t) = 0$$

Numerical Procedure

Space and time derivatives in Eqs. (1) and (2) were replaced by central differences in the interior of a finite-difference mesh. Forward or backward differences, as appropriate, were used at the shell apex or edge. If the values of normal and tangential displacements in the mesh are denoted by w_i^j and u_i^j, where i refers to a space index and j to time index, we find the following relations:

$$\begin{aligned} w_i^{j+1} = (aw_{i+2} + bw_{i+1} + cw_i + dw_{i-1} + ew_{i-2})^j + \\ (fu_{i+2} + gu_{i+1} + hu_i + ku_{i-1} + lu_{i-2})^j + \\ mw_i^{j-1} + np_i^j \quad (3) \end{aligned}$$

$$\begin{aligned} u_i^{j+1} = (qu_{i+1} + ru_i + su_{i-1})^j + tu_i^{j-1} + \\ (\alpha w_{i+2} + \beta w_{i+1} + \gamma w_i + \delta w_{i-1} + \epsilon w_{i-2})^j \quad (4) \end{aligned}$$

The coefficients in Eqs. (3) and (4) are easily obtained directly from Eqs. (2) and (1) and are not reported here in the interest of brevity.

Clearly, such a numerical procedure is simplicity itself in outline. If u and w have been determined at all space points i on two time line j and $j - 1$, one can directly calculate the solution in time. The solution values required to start the whole process are obtained from initial conditions. The simplicity of the procedure is reflected in significant advantages for computation speed, ease of programing, and memory requirements. It should also be pointed out that, for such an explicit procedure, nonlinear problems are no more difficult than linear problems.

Such explicit finite-difference procedures are, of course, limited by questions of numerical instability. A thorough analysis yielding stability criteria has been obtained for only a very limited class of linear differential equations. For nonlinear equations with the complexity of Eqs. (1) and (2), such an analysis would certainly be difficult, if not impossible. We content ourselves with a more pragmatic approach based on results of numerical experimentation. It was found that, if the space increment $\Delta s = R\Delta\phi$ and time increment Δt were chosen so as to satisfy

$$\Delta t < \Delta s(\rho/E)^{1/2} \quad (5)$$

then computations exhibited numerical stability for all cases attempted. These included a variety of time-dependent loadings, of boundary conditions, as well as of basic geometry (e.g., conical and cylindrical shells were also examined using

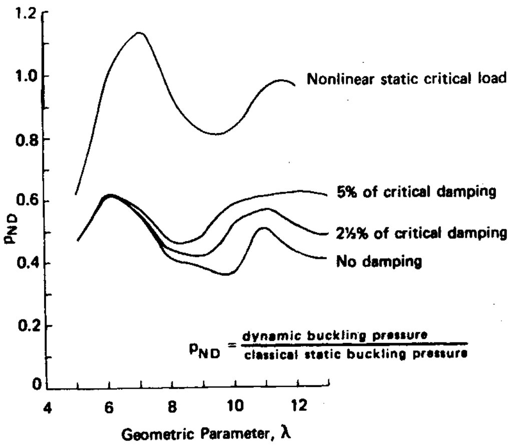

Fig. 2 Buckling pressure vs geometric parameter for several values of damping.

the same procedure). On the other hand, if the inequality were violated by even a small amount, numerical instability quickly and dramatically evidenced itself. Equation (5) is the same as employed by Johnson and Greif[9] in a comparison of implicit and explicit procedures for linear cylindrical shell problems, and by Leech, Witmer, and Pian[10] in their very general approach.

The accuracy of the numerical results was also established in a pragmatic way. It was found that using 40 mesh points along the middle surface of the shell provided a solution that generally was not changed significantly (less than a percent) when double this number of mesh points was used. A comparison of these results with available and reliable results provided further assurance of a sufficient degree of accuracy. Finally, a comparison of the predicted buckling pressure with the extremely sparse experimental data available showed satisfactory agreement.[11]

In connection with the matter of accuracy, it is worth noting that the governing equations are very sensitive to higher-order terms when the values of applied pressure are close to the buckling value. If N_ϕ in Eq. (2) is expressed in terms of β, there is a temptation to neglect terms $O(\beta^3)$. Such a practice can lead to the gradual emergence of excessively large values of deflection which can be misinterpreted as buckling or even as evidence of numerical instability that is not really present. This sensitivity was found for both explicit and implicit procedures, although it affects the explicit method more adversely. Stricklin and Martinez[4] also encounter this situation in a finite-element procedure.

Comparison of Explicit and Implicit Procedures

Because the program behind this study envisioned a wide variety of geometries and loadings, it was deemed advisable to ascertain quantitatively the advantages of the explicit procedure over the implicit. In this connection, then, a computer program was written based on an implicit approach to the same problem and served as a vehicle for comparison. It was found that running times for the implicit approach were three times longer, that programing the explicit procedure was vastly simpler, and that the memory required was significantly less for the explicit approach. The stability considerations for the explicit procedure were easily resolved. Furthermore, convergence problems associated with the nonlinear character of the equations emerged with the implicit procedure when the load is close to the buckling pressure. This fact is in contrast to the unconditionally stable nature of the implicit procedure when the equations are linear.[12] Johnson and Greif[9] and Rossettos and Parisse[13] found that, in the linear case, when the time increments selected were larger than $\frac{1}{50}$ of the period of a given mode of vibration, the implicit procedure would have a damping effect on the response. This phenomenon was found not true in the nonlinear case.

Numerical Results

The problem of a clamped spherical shell under a suddenly applied and uniformly distributed pressure loading was considered with and without damping terms. Values of the dynamic buckling load were determined as a function of the geometric parameter, $\lambda^2 = m^2\phi_0{}^2R/h$, where $m^4 = 12(1 - \nu^2)$. [See Fig. 2, where we plot dynamic buckling pressure normalized by the classical static value for a complete sphere, i.e., $P = 4Eh^2/(R^2m^2)$.] The criterion used to determine the dynamic buckling pressure is that first proposed by Budiansky and Roth.[14] It is based on the observation that, if the maximum value of the average deflection $\bar{\rho}(t)$, where

$$\bar{\rho}(t) \equiv K \int_0^{r_0} wr\, dr \qquad K = \frac{2m^2}{(hR^2\phi_0{}^2)}$$

is plotted vs the intensity of applied pressure, one finds a very sudden (nearly discontinuous) change in the curve at or near the buckling pressure. This criterion was also used in Refs. 1–4. It was found in the present study, as in these references, that the peak value of $\bar{\rho}$ may occur after several cycles, so that a determination of critical pressure based on observation of only one or two oscillations may result in overestimating p_{cr}. In the present study, at least 15 oscillations (sometimes more) were observed before deciding that the pressure was below critical. (See Fig. 3 for typical results.) The presence of late response buckling loads has been observed experimentally.[11]

A. Without Damping

In the case of zero damping, the results of the present study are in good agreement with the results of Huang,[2] Stephen and Fulton,[3] and Stricklin and Martinez.[4] These results were all obtained by independent numerical procedures, and the degree of their agreement may be taken as an indication of a high level of accuracy. This is a relevant point in view of a) the significant difference between these results and those of earlier, more approximate techniques, and b) the lack of any extensive and detailed experimental data. For example, the careful experiments of Lock et al.[11] concerned only the single geometric parameter $\lambda = 7.5$. Their estimate of critical dynamic pressure for the value of λ agrees well with that reported here and in Refs. 2–4.

B. Influence of Damping

To study the role played by velocity damping in the buckling of spherical shells, we first define a scale to quantify the amount of damping present in terms of the critical value of the damping parameter $\bar{\lambda}$. $\bar{\lambda}_{cr}$, then, is the minimum value of $\bar{\lambda}$ for which a dynamically excited shell monotonically moves to its static equilibrium position. This parameter de-

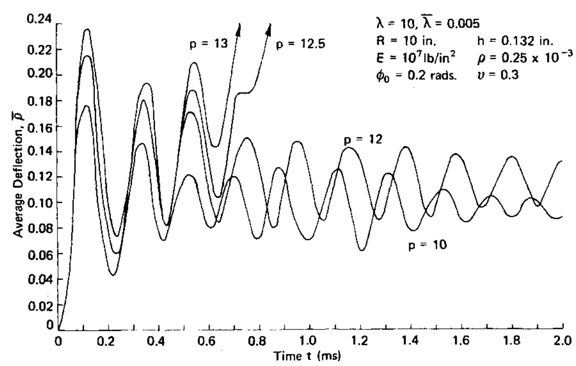

Fig. 3 Representative $\bar{\rho}$ vs t curves.

pends upon the shell geometry, the material constants, and, for the nonlinear case, upon the magnitude of the load. It was determined numerically that $\bar{\lambda}_{cr}$ scales directly with the elastic modulus and, for a given geometry, is nearly insensitive to the magnitude of applied pressure even for values of pressure quite close to the buckling load for the undamped shell. This was found to be so for the step loading in time, both for spatially uniform pressures and for pressures distributed over a portion of the shell surface. It was also found numerically that $\bar{\lambda}_{cr}$ is only moderately sensitive to the geometric parameter λ. See, for example, Fig. 4, where we plot the response of the apex for a fixed value of $\bar{\lambda} = 0.05$ for several values of λ. In each plot, the value of pressure (applied uniformly in space, stepwise in time) is approximately one-half of the critical dynamic pressure for the associated λ. In determining $\bar{\lambda}_{cr}$, it was verified by an independent calculation that the equilibrium position attained did, in fact, correspond to the prediction of a nonlinear static analysis[15] (see Fig. 5). In this way, one also establishes that for $\bar{\lambda} \geq \bar{\lambda}_{cr}$ the dynamic buckling pressure is the same as in the static case. In Fig. 5, the case $p = 19$ corresponds to a load above the static buckling load.

Realistic estimates of the amount of damping one might reasonably expect to find in practice range from 1% to 5% of critical damping. Values of critical pressure levels as a function of the geometric parameter λ, with $\bar{\lambda}$ equal to 2.5% of critical and 5% of critical, are plotted in Fig. 2. It may be seen that, in contrast to the results of Hegemier and Tzung,[7] there is no discontinuity in the behavior of the buckling of a spherical cap introduced by the presence of small or even relatively large amounts (5%) of damping. The dynamic buckling loads for this loading and zero damping are on the order of $\frac{1}{2}$ (or less) the static values. As damping increases toward $\bar{\lambda}_{cr}$, one approaches the static value smoothly. This was found to be the case over the range of geometric parameter λ studied, which includes most of the range of practical interest.

Summary

The explicit numerical procedure outlined has been directly compared with implicit methods and found to be highly efficient. Reductions in computer time to one-third of that required by implicit methods are representative. This is significant in nonlinear dynamic problems that may require determination of solutions at hundreds (even thousands) of time steps and, thus, orders of magnitude more computations than their static counterparts. The trend in the literature has been to use implicit methods exclusively.[1–3] The accuracy of the explicit method has been established internally by reducing mesh spacing, and externally by comparison with independent solutions and with experiment.

A rather useful property of a numerical procedure is flexibility. In this context, brief studies were made for a variety of geometrical shapes (cones and cylinders), boundary conditions, and spatial and temporal distributions of applied loading. For all these cases, the explicit procedure was found

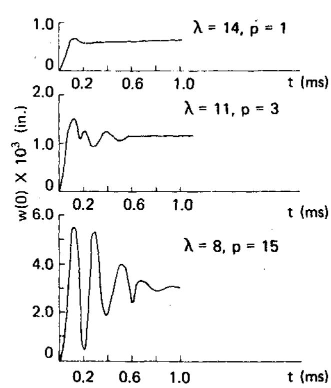

Fig. 4 Apex response for step pressure load (constant damping, $\bar{\lambda}$ = 0.05, several geometric parameters).

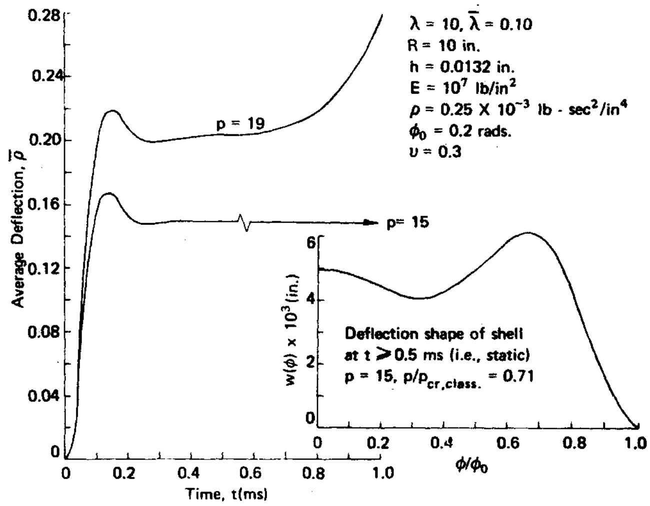

Fig. 5 Average deflection vs time.

to work well. Perhaps the most interesting variation on loading considered was one in which a sinusoidal pressure is superimposed on uniform one

$$p = p_0 + p_1 \cos(\omega t + \delta) \qquad (6)$$

Spherical shells with this loading were studied theoretically and experimentally by Evan-Iwanowski.[16] Details are sufficiently intricate to preclude lengthy discussion here and will be reserved for a later report. We merely note that results were obtained in excellent agreement with experimental values.

Finally, it is worth noting that the same numerical procedure described here can be used in a one-dimensional treatment of stress waves in elastic rods. Its use in such problems should be very effective, since the time scales are usually much shorter than those in the problem of the present paper.

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Large Strain, Elasto-Plastic Finite Element Analysis

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A method is presented for the large strain, elasto-plastic analysis of two-dimensional structures by the finite element method. An incremental variational principle is used to develop the finite element equilibrium equations for use in a piecewise linear solution procedure. The present formulation differs from other papers in that the variational principle leads to a set of equations which include an equilibrium check. The equilibrium check can be employed at any point in the incremental solution process, with a Newton-Raphson iteration, to reduce any cummulative error in nodal point equilibrium. Such errors are caused by linearizing the displacement equilibrium equations, and can build up to such an extent that the computed results are meaningless. The equilibrium check and corrective cycling procedure presented herein prevent the computed load-deflection behavior from straying from the true equilibrium path, and they represent a major contribution of this paper.

Nomenclature

A = cross-sectional area of truss member
a_s, a_o = initial and final inner radius, respectively, of folded sheet
C_{ijkl} = constitutive operator
E, E_p = elastic and postyield moduli, respectively
ΔE_{ij} = incremental Green's strain tensor
$\{E\}$ = vector of residual error forces
e = strain at inner radius of folded sheet
$\{\Delta F\}$ = vector of incremental forces
Δf_i = component of element force vector
k_{ij} = elements of incremental stiffness matrix
$k_{ij}^{(G)}$ = elements of geometric stiffness matrix
$[K]_N$ = system stiffness matrix at step N
$[K^{(G)}]_N$ = system geometric stiffness matrix at step N
L = length of truss member
N = load step N
q = applied pressure load
$\{\Delta R\}$ = incremental system displacements
Δr_i = incremental elemental displacement
s = surface
t = thickness of bent sheet
$\Delta T_i, T_i, T_i^{(0)}$ = incremental, total and initial surface tractions, respectively
Δu_i = incremental displacement
v = volume
X_i = local (initial) coordinates
x_i = current coordinates
δ_{ij} = Kronecker delta
$\delta(\ )$ = variational operator
ϵ = error measure
ϵ_i = error force at element coordinate i
$\Delta\epsilon_{ij}$ = linear portion of strains
$\Delta\eta_{ij}$ = nonlinear strains
ϕ_{ij} = shape functions
$\Delta\sigma_{ij}, \sigma_{ij}, \sigma_{ij}^{(0)}$ = incremental, total and initial stress tensors
σ_y = yield stress
μ = Poisson's ratio
$\Delta\omega_{ij}$ = incremental rotation tensor

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Introduction

MANY authors have applied the finite element method to geometrically nonlinear problems.[1–8] From this work, basically three classes of finite element formulations can be defined: Class 1) incremental methods without equilibrium checks; Class II) direct solutions of the governing nonlinear equations; and Class III) incremental methods with equilibrium checks.

Historically, Class I was the first finite element approach to solving geometrically nonlinear problems.[1,8–10] In this method, the load is applied to the structure in small increments, and the incremental displacements due to each load step are determined. Incremental stresses and strains are computed at each step and used in the following load step. Although this method is computationally very fast, it has the disadvantage that equilibrium at any particular load level is not necessarily satisfied. No attempt is made to determine if equilibrium requirements are indeed met, and the same problem must be repeatedly solved using successively smaller load steps to assess the solution accuracy.

The direct solution method (Class II) involves applying the total load to the structure and computing the total response by using mathematical iterative techniques. This approach may be subdivided into two distinct categories; a) direct minimization of the potential energy, and b) direct solution of the nonlinear algebraic equilibrium equations.